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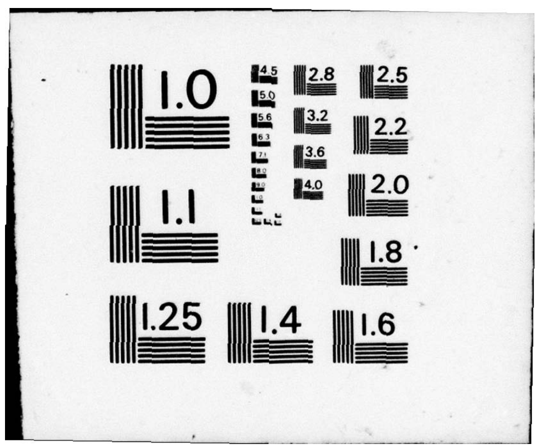
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Solving Queues without Rouché's Theorem

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Abstract

A novel analysis of the steady-state probabilities of a class of infinite Markov chains is given. Markov chains of this type appear in the study of bulk queues and a variety of other stochastic models. Algorithms, which involve only real arithmetic and avoid the traditional analysis, based on Rouché's theorem, are presented.

Key Words

Markov chain, Queueing theory, steady-state probabilities, computational probability, Bailey's bulk queue, Moran dam

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1. Introduction

There is a long tradition in queueing theory of presenting the probability distributions of interest in the form of Laplace-Stieltjes, Fourier or generating function transforms. These transforms often involve several unknown parameters, which need to be evaluated by the solution of an auxiliary system of linear equations with complex coefficients. The coefficients of that auxiliary system depend, in addition, on the roots of a (usually) transcendental equation inside some region of the complex plane. The numerical computation of the probability distributions of interest, by this classical method based on an application of Rouché's theorem, is in practice quite difficult. It can also be numerically hazardous, when the roots of the transcendental equation lie close together or coincide. From an aesthetic viewpoint it is unattractive, as it involves several steps without a clear probabilistic significance. In spite of its use in literally hundreds of papers on queues, this method has also been largely ignored by the practitioner who attacks the numerics of queueing problems by ad hoc truncation procedures or by computer simulation.

It is possible however to show that many queueing problems can be solved by a purely probabilistic method which requires little or no complex analysis. This approach leads, at least in the range of practical interest, to efficient and stable algorithms involving only real arithmetic. The theoretical background material for this procedure is discussed in Neuts [8, 11] and involves some elementary functional analysis. Applications to the queue with semi-Markov service times and to an M/G/1 queue with several types of customers are available in Neuts [12, 13]. Extensive numerical examples and computer programs are appended to the preprint of the paper [13].

In the present paper, which is partly expository, we shall begin by giving a detailed discussion of the class of Markov chains with a transition probabil-

ity matrix P of the form given in Formula (1). Chains of this type arise frequently in queueing models. In the latter part of the paper, we shall show how a large variety of explicit or computationally tractable new results can be obtained for some specific models, discussed in the literature.

We consider Markov chains with transition probability matrix P, given by

$$(1) \qquad P = \begin{vmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{m-1,0} & b_{m-1,1} & b_{m-1,2} & b_{m-1,3} & b_{m-1,4} & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix},$$

where the sequences $\{a_\nu\}$ and $\{b_{k\nu}\}$, $0 \leq k \leq m-1$, are (discrete) probability densities with finite means α^* and β_k^*, $0 \leq k \leq m-1$, respectively. Their probability generating functions, used only for analytic convenience, will be denoted by $A^*(z)$ and $B_k^*(z)$, $0 \leq k \leq m-1$, respectively and in order to avoid triviality, we assume that $m \geq 1$.

A. Irreducibility and Aperiodicity

In applications, it is usually obvious from the specific form of the entries that the chain P is irreducible and aperiodic, but a general discussion of these properties is tedious and not very interesting. It is easy to see

that under the two conditions, stated below, the chain is both irreducible and aperiodic.

Condition 1. Let $d = \text{g.c.d}\ \{k: a_k > 0,\ k \geq 1\}$, then we assume that

$$\text{g.c.d.}(m,d) = 1, \text{ and } a_0 > 0. \tag{2}$$

Condition 2. Let the matrix B_0 be defined by

$$B_0 = \begin{vmatrix} b_{00} & \cdots & b_{0,m-1} \\ \vdots & & \\ b_{m-1,0} & \cdots & b_{m-1,m-1} \end{vmatrix}, \tag{3}$$

then we assume that the matrix $I-B_0$ is nonsingular.

Remark

Condition 2 guarantees that there is no recurrent class contained in $\{0, 1, \ldots, m-1\}$. Condition 1 implies that there is positive probability of entering the set $\{0, 1, \ldots, m-1\}$ from any state $j \geq m$ and that the possible times of such entries are aperiodic. We shall assume henceforth that Conditions 1 and 2 hold, so that the chain P is both irreducible and aperiodic.

B. A Partition of the Matrix P

We partition the matrix P in m x m blocks in the natural manner to obtain

$$P = \begin{vmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ 0 & 0 & 0 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}, \tag{4}$$

where

$$(5)\qquad B_j = \begin{vmatrix} b_{0,mj} & \cdots & b_{0,mj+m-1} \\ \vdots & & \vdots \\ b_{m-1,mj} & \cdots & b_{m-1,mj+m-1} \end{vmatrix}, \qquad \text{for } j \geq 0,$$

and

$$A_j = \begin{vmatrix} a_{mj} & \cdots & a_{mj+m-1} \\ \vdots & & \vdots \\ a_{mj-m+1} & \cdots & a_{mj} \end{vmatrix}, \qquad \text{for } j \geq 1,$$

$$A_0 = \begin{vmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & a_0 & \cdots & a_{m-2} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_0 \end{vmatrix}.$$

Much of the general theory remains valid for stochastic matrices, which can be written in the partitioned form (4), but in the present case the special form of the matrices A_j, $j \geq 0$, leads to some simplifying properties, noted in Lemma 1 below.

We first introduce some notation. The stochastic matrices A and B are defined by

$$(6)\qquad A = \sum_{j=0}^{\infty} A_j, \qquad B = \sum_{j=0}^{\infty} B_j.$$

Furthermore, with $\underline{e} = (1, 1, \ldots, 1)'$, we define

$$(7)\qquad \underline{\alpha}^0 = \sum_{j=1}^{\infty} j\, A_j\, \underline{e}, \qquad \underline{\beta}^0 = \sum_{j=1}^{\infty} j\, B_j\, \underline{e},$$

and we denote the invariant probability vector of A by $\underline{\pi}$.

Lemma 1

The matrix A is a circulant and

(8) $$\underline{\pi} = \frac{1}{m}\underline{e}'.$$

If U is an integer-valued random variable with density $\{a_j\}$, then

(9) $$\alpha_k^0 = E[\frac{U+k-1}{m}], \qquad \text{for } 1 \leq k \leq m,$$

where $[\cdot]$ is the integer part function. Furthermore the inner product $\underline{\pi}\,\underline{\alpha}^0$ is given by

(10) $$\underline{\pi}\,\underline{\alpha}^0 = m^{-1}\alpha^*.$$

Proof.

It is clear that A is a circulant and this implies (8). See e.g. Rosenblatt [16] p. 52. Furthermore

(11) $$\alpha_k^0 = \sum_{\nu=1}^{\infty} \nu \sum_{r=0}^{m-1} a_{m\nu-k+r+1} = E[\frac{U+k-1}{m}],$$

for $1 \leq k \leq m$. For every integer $N \geq 0$, we have the elementary equality

(12) $$\sum_{j=1}^{m} [\frac{N+j-1}{m}] = N,$$

which clearly implies (10).

Henceforth we shall refer to the set of states $\{mr, mr+1, \ldots, mr+m-1\}$ as level r, for $r \geq 0$. For convenience we shall refer to the original state $mr+j-1$ also as state (r,j), for $r \geq 0$, $1 \leq j \leq m$.

2. A First Passage Problem

In [8], we examined the first passage from the state $(r+1, j)$ to the level r. By $G_{jj'}(k)$, $k \geq 1$, we denoted the probability that, starting in the

state $(r+1,j)$, the level r is reached for the first time after exactly k transitions by a visit to the state (r,j'). The sequence of matrices $G(k) = \{G_{jj'}(k)\}$, $k \geq 1$, is of basic importance. We showed that its matrix generating function

$$\hat{G}(z) = \sum_{k=1}^{\infty} G(k) z^k, \tag{13}$$

satisfies the matrix functional equation

$$\hat{G}(z) = \sum_{\nu=0}^{\infty} z A_\nu \hat{G}^\nu(z), \qquad \text{for } |z| \leq 1. \tag{14}$$

In an appropriately defined set of matrices $\hat{G}(z)$ is also the unique solution to Equation (14) for $0 \leq z < 1$. The following theorems summarize results proved in [8] and [11].

<u>Theorem 1</u>

Defining the corresponding first passage probability matrices $G^{(i)}(k)$, $k \geq i$, between level $r+i$ and level r, the matrix generating function of the sequence $\{G^{(i)}(k)\}$, is given by the i-th power $\hat{G}^i(z)$ of $\hat{G}(z)$, for $i \geq 1$.

The matrix G, defined by

$$G = \sum_{k=1}^{\infty} G(k) = \lim_{z \to 1-} \hat{G}(z), \tag{15}$$

is <u>stochastic</u> if and only if

$$\rho = \underline{\pi}\, \underline{\alpha}^0 = m^{-1} \alpha^* \leq 1. \tag{16}$$

The matrix G is then the unique solution of the equation

$$G = \sum_{\nu=0}^{\infty} A_\nu G^\nu, \tag{17}$$

in the set of substochastic matrices. The minimal solution $\hat{G}$ to (17) in the

set of substochastic matrices is strictly positive. If $\rho > 1$, $\hat{G}$ is strictly substochastic and for $\rho \leq 1$, we have that $\hat{G} = G$ is stochastic.

Remarks

a. The numerical solution to (17) was examined in [11]. For most practical purposes, G may be calculated efficiently by successive substitutions in the modified form

$$G = (I - A_1)^{-1} \sum_{\substack{\nu=0 \\ \nu \neq 1}}^{\infty} A_\nu G^\nu, \tag{18}$$

of Equation (17), starting with $G = 0$.

b. The special structure of the matrices A_ν, $\nu \geq 0$, does not appear to lead to any tractable special properties of the matrix G. In particular G is not a circulant. This is shown by the example, where $m = 2$ and

$$A_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & 0 \end{pmatrix},$$

and $A_\nu = 0$, for $\nu \geq 3$. This rare tractable example has the solution matrix G, given by

$$G = \begin{pmatrix} \frac{1}{2}(\sqrt{17} - 3) & \frac{1}{2}(5 - \sqrt{17}) \\ 2\sqrt{17} - 8 & 9 - 2\sqrt{17} \end{pmatrix}.$$

For $\rho \leq 1$, the stochastic matrix G has an invariant probability vector $\underline{g}$, which satisfies $\underline{g} = \underline{g}\, G$, and $\underline{g}\, \underline{e} = 1$. By $\tilde{G}$ we denote the $m \times m$ matrix with m identical rows given by $\underline{g}$. It is well-known that G and $\tilde{G}$ commute and that the matrix $I - G + \tilde{G}$ is nonsingular.

We define the matrix M by

$$M = \hat{G}'(1-) = \sum_{\nu=1}^{\infty} \nu G(\nu), \tag{19}$$

and the vector $\underline{\mu}$ by $\underline{\mu} = M\,\underline{e}$. The matrix $\Delta(\underline{\alpha}^0)$ is the diagonal matrix $\mathrm{diag}(\alpha_1^0, \ldots, \alpha_m^0)$. The following useful results were established in [11].

Theorem 2

When $\rho = 1$, the matrix M is infinite. When $\rho < 1$, the matrix M is the solution of the equation

$$M = G + \sum_{i=1}^{\infty} A_i \sum_{\nu=0}^{i-1} G^{\nu} M\, G^{i-\nu-1}. \tag{20}$$

The vector $\underline{\mu}$ satisfies

$$\underline{\mu} = \underline{e} + \sum_{i=1}^{\infty} A_i \sum_{\nu=0}^{i-1} G^{\nu} \underline{\mu}, \tag{21}$$

and is given explicitly by

$$\underline{\mu} = (I-G+\tilde{G})[I-A+\tilde{G}-\Delta(\underline{\alpha}^0)\tilde{G}]^{-1}\underline{e}. \tag{22}$$

The scalar product $\underline{g}\,\underline{\mu}$ is given by

$$\underline{g}\,\underline{\mu} = (1-\rho)^{-1} = m(m-\alpha^*)^{-1}. \tag{23}$$

Remarks

a. In the sequel we shall express a large number of features of the Markov chain P in terms of the matrix G and the vectors $\underline{g}$ and $\underline{\mu}$. These entities are the crucial elements to be evaluated in a numerical solution of these features. The equation (23) is a powerful check on the numerical accuracy of the computed values of $\underline{g}$ and $\underline{\mu}$.

b. The entries of the matrices G and M and of various other matrices expressible in terms of these have a number of interesting interpretations in the queueing problems, which lead to Markov chains of the type P. Some of these interpretations will be discussed below.

3. The Invariant Probability Vector of the Matrix P

The invariant probability vector $\underline{x}$ of the matrix P satisfies the stationarity equations

$$(24) \qquad x_j = \sum_{\nu=0}^{m-1} x_\nu b_{\nu j} + \sum_{k=0}^{j} x_{m+j-k} a_k, \qquad \text{for } j \geq 0.$$

The probability generating function

$$(25) \qquad X(z) = \sum_{k=0}^{\infty} x_k z^k, \qquad \text{for } |z| \leq 1,$$

is given by

$$(26) \qquad X(z) = \frac{\sum_{\nu=0}^{m-1} x_\nu z^\nu [A^*(z) - z^{m-\nu} B_\nu^*(z)]}{A^*(z) - z^m}.$$

We see that the right-hand side depends on the components of the vector $\underline{\xi} = (x_0, x_1, \ldots, x_{m-1})$, which we shall now determine by a probabilistic argument.

The matrix K is defined by

$$(27) \qquad K = \sum_{\nu=0}^{\infty} B_\nu G^\nu,$$

and is clearly stochastic.

Lemma 2

The matrix K is irreducible.

Proof

If K were reducible, its rows and columns could be permuted in the same way, so that K could be written in the form

$$(28) \qquad K = \begin{pmatrix} K_1 & K_2 \\ 0 & K_3 \end{pmatrix}.$$

This implies that the matrices B_0 and B_ν, for $\nu \geq 1$, must be of the form

(29) $$B_0 = \begin{pmatrix} B_0(1) & B_0(2) \\ 0 & B_0(3) \end{pmatrix}, \qquad B_\nu = \begin{pmatrix} B_\nu(1) & B_\nu(2) \\ 0 & 0 \end{pmatrix}, \text{ for } \nu \geq 1,$$

since the matrix G is strictly positive.

This further shows that $B_0(3)$ is stochastic, and therefore that $I - B_0$ is singular. This contradicts Condition 2, which we assumed to hold.

The invariant probability vector $\underline{\gamma}$ of K is therefore well-defined and satisfies

(30) $$\underline{\gamma} = \underline{\gamma}K, \qquad \underline{\gamma}\,\underline{e} = 1.$$

We further define the vector $\underline{\theta}$ by

(31) $$\underline{\theta} = \underline{e} + \sum_{\nu=1}^{\infty} B_\nu \sum_{j=0}^{\nu-1} G^j \underline{\mu},$$

where G and $\underline{\mu}$ are as defined in Section 2.

Lemma 3

The vector $\underline{\theta}$ is given explicitly by

(32) $$\underline{\theta} = \underline{e} + (1-\rho)^{-1}\underline{\beta}^0 + (B-K)(I-G+\tilde{G})^{-1}\underline{\mu}.$$

Proof

Since $G^\nu(I-G+\tilde{G}) = G^\nu - G^{\nu+1} + \tilde{G}$, for $\nu \geq 0$, we obtain

(33) $$\sum_{\nu=1}^{\infty} B_\nu \sum_{j=0}^{\nu-1} G^j (I-G+\tilde{G}) = B - K + \Delta(\underline{\beta}^0)\tilde{G},$$

where $\Delta(\underline{\beta}^0) = \mathrm{diag}(\beta_1^0,\ldots,\beta_m^0)$. Since $I - G + \tilde{G}$ is nonsingular we obtain

(34) $$\begin{aligned}\underline{\theta} &= \underline{e} + [B-K+\Delta(\underline{\beta}^0)\tilde{G}](I-G+\tilde{G})^{-1}\underline{\mu} \\ &= \underline{e} + (1-\rho)^{-1}\underline{\beta}^0 + (B-K)(I-G+\tilde{G})^{-1}\underline{\mu},\end{aligned}$$

by noting that $\Delta(\underline{\beta}^0)\tilde{G}\,\underline{\mu} = (\underline{g}\,\underline{\mu})\underline{\beta}^0 = (1-\rho)^{-1}\underline{\beta}^0$.

Corollary 1

The scalar product $\underline{\gamma}\,\underline{\theta}$ is given by

$$\underline{\gamma}\,\underline{\theta} = 1 + (1-\rho)^{-1}\underline{\gamma}\,\underline{\beta}^0 - \underline{\gamma}(I-B)(I-G+\tilde{G})^{-1}\underline{\mu}. \tag{35}$$

Proof

From Formula (32) and the definition of $\underline{\gamma}$.

Theorem 3

The vector $\underline{\xi} = (x_0,\ldots,x_{m-1})$ is given by

$$\underline{\xi} = (\underline{\gamma}\,\underline{\theta})^{-1}\underline{\gamma}. \tag{36}$$

Proof

We consider the successive visits to the set of states $\{0,1,\ldots,m-1\}$ in the Markov chain P. If J_n denotes the state at the n-th visit and X_n the time between the (n-1)st and the n-th visit, then the sequence of pairs $\{(J_n,X_n),\ n \geq 0\}$ with $X_0 = 0$, is easily seen to be a Markov renewal sequence with m states. The transition probability matrix is of lattice type. An easy first passage argument shows that the probability generating matrix function K(z) of that transition probability matrix is given by

$$K(z) = \sum_{\nu=0}^{\infty} z\, B_\nu \hat{G}^\nu(z), \qquad \text{for } |z| \leq 1. \tag{37}$$

We clearly have that the matrix K, defined in (27), and the vector $\underline{\theta}$, defined in (31), satisfy

$$K = K(1-), \qquad \underline{\theta} = K'(1-)\underline{e}. \tag{38}$$

Application of a classical theorem on Markov renewal processes, see e.g.

Çinlar [2], Thm. 6.12, p. 155, or Hunter [4], Thm. 2.11, p. 196, yields that the mean recurrence time E_j of the state $(0,j)$ in the finite Markov renewal process is given by

$$E_j = (\underline{\gamma}\,\underline{\theta})\gamma_j^{-1}, \qquad \text{for } 1 \le j \le m. \tag{39}$$

Since E_j is also the expected number of transitions between successive returns to the states $j-1$ in the Markov chain P, we have that

$$E_j = (x_{j-1})^{-1}, \qquad \text{for } 1 \le j \le m, \tag{40}$$

but this is equivalent to (36).

<u>Corollary 2</u>

$$\sum_{\nu=0}^{m-1} x_\nu = (\underline{\gamma}\,\underline{\theta})^{-1}, \tag{41}$$

and the right-hand side is explicitly given by applying Formula (35).

The higher terms x_j, $j \ge m$, of the stationary probability density may be computed recursively once the initial terms $x_0, x_1, \ldots, x_{m-1}$, are known. From the stationarity equations (24), we obtain

$$x_{m+j} = a_0^{-1}[x_j - \sum_{\nu=0}^{m-1} x_\nu b_{\nu j} - \sum_{k=1}^{j} x_{m-k+j}\, a_k], \qquad \text{for } j \ge 0. \tag{42}$$

On computers with a short word length, the recurrence relation (42) is for large values of j quite sensitive to loss of significance, because there are a substantial number of small terms to be subtracted from the small positive value of x_j. For the case of the M/G/1 queue, Dr. Paul J. Burke suggested to the author an alternate form [10] of the recursion formula for the x_j, which is far less sensitive to loss of signifcance. For the Markov chain P, this alternate form is given by the following result.

Theorem 4

The quantities x_j, for $j \geq m$, may be recursively computed by use of the formulas

$$(43)\qquad x_m = a_0^{-1}\left[\sum_{\nu=0}^{m-1} x_\nu \hat{b}_{\nu 0} - \sum_{\nu=0}^{m-1} x_\nu\right],$$

$$x_{m+j} = a_0^{-1}\Big\{\sum_{\nu=1}^{j} x_{m+j-\nu}\hat{a}_\nu + \sum_{\nu=0}^{m-1} x_\nu \hat{b}_{\nu j}$$

$$- \Big[\sum_{\nu=j+1}^{m-1} x_\nu + \sum_{\nu=0}^{j-1} x_{m+\nu}\Big]\Big\}, \quad \text{for } 1 \leq j \leq m-2,$$

$$x_{m+j} = a_0^{-1}\left[\sum_{\nu=1}^{j} x_{m+j-\nu}\hat{a}_\nu + \sum_{\nu=0}^{m-1} x_\nu \hat{b}_{\nu j} - \sum_{\nu=1}^{m-1} x_{m+j-\nu}\right], \quad \text{for } j \geq m+1,$$

where

$$(44)\qquad \hat{b}_{\nu j} = 1 - \sum_{k=0}^{j} b_{\nu k}, \qquad \text{for } 0 \leq \nu \leq m-1,\ j \geq 0,$$

$$\hat{a}_j = 1 - \sum_{k=0}^{j} a_k, \qquad \text{for } j \geq 0.$$

Proof

The generating function $X_1(z) = z^{-m}[X(z) - \sum_{\nu=0}^{m-1} x_\nu z^\nu]$, satisfies

$$(45)\qquad z^m X_1(z) + \sum_{\nu=0}^{m-1} x_\nu z^\nu = \sum_{\nu=0}^{m-1} x_\nu B_\nu^*(z) + A^*(z)X_1(z),$$

and this leads to

$$(46)\qquad \frac{1-z^m}{1-z}X_1(z) = \frac{1-A^*(z)}{1-z}X_1(z) + \sum_{\nu=0}^{m-1} x_\nu \frac{1-B_\nu^*(z)}{1-z} - \sum_{\nu=1}^{m-1} x_\nu \frac{1-z^\nu}{1-z}.$$

Recalling that

$$(47)\qquad \frac{1-A^*(z)}{1-z} = \sum_{k=0}^{\infty} \hat{a}_k z^k, \qquad \frac{1-B_\nu^*(z)}{1-z} = \sum_{k=0}^{\infty} \hat{b}_{\nu k} z^k,$$

for $0 \leq \nu \leq m-1$, and expanding both sides in (46), we obtain upon equating the coefficients of z^j, the expressions given in Formula (43).

[redacted]

Corollary 3

The equality

$$(m-\alpha^*)[1 - \sum_{\nu=0}^{m-1} x_\nu] = \sum_{\nu=0}^{m-1} x_\nu(\beta^*-\nu), \tag{48}$$

holds.

Proof

By letting $z \to 1-$ in (46) and applying Abel's theorem.

Corollary 4

The mean $\tilde{L}$ of the probability density $\{x_j\}$ is given by

$$\tilde{L} = \frac{1}{2}(m-\alpha^*)^{-1}\{[\alpha^*(2)-m(m-1)+2m(m-\alpha^*)][1 - \sum_{\nu=0}^{m-1} x_\nu] + \sum_{\nu=0}^{m-1} x_\nu[\beta^*_\nu(2)-\nu(\nu-1)+2\nu(m-\alpha^*)]\}, \tag{49}$$

where $\alpha^*(2)$ and $\beta^*_\nu(2)$, $0 \leq \nu \leq m-1$, are the second factorial moments of the densities $\{a_j\}$ and $\{b_{\nu j}\}$, $0 \leq \nu \leq m-1$, respectively.

Proof

By differentiating in Formula (26) and using de l'Hôpital's rule, we obtain

$$2(m-\alpha^*)\tilde{L} = [\alpha^*(2) - m(m-1)][1 - \sum_{\nu=0}^{m-1} x_\nu] + \sum_{\nu=0}^{m-1} \beta^*_\nu(2)x_\nu - \sum_{\nu=0}^{m-1} \nu(\nu-1)x_\nu + 2m \sum_{\nu=0}^{m-1} \beta^*_\nu x_\nu - 2\alpha^* \sum_{\nu=0}^{m-1} \nu x_\nu, \tag{50}$$

and upon replacing $\sum_{\nu=0}^{m-1} \beta^*_\nu x_\nu$ by the expression, obtained from (48), we find the expression given in (49).

4. Bailey's Bulk Queue

One of the earliest papers to use the Rouché method of solution was N. T. J. Bailey's treatment [1] of a bulk queueing model, involving a server who becomes available at the epochs of a renewal process with underlying distribution $H(\cdot)$. Customers arrive according to a Poisson process of rate λ. If k customers are present when the server becomes available, a group of size $\min(k,m)$ enters service.

It is well-known that the successive queue lengths immediately prior to the beginnings of services form a Markov chain of the type (1) in which $b_{\nu j} = a_j$, for $0 \leq \nu \leq m-1$, and $j \geq 0$. The quantities a_j are given by

$$a_j = \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^j}{j!} \, d\,H(u), \qquad \text{for } j \geq 0. \tag{51}$$

We shall not concern ourselves with the particular form (51), but it is interesting to note the consequences of the fact that the first m rows are identical and given by $\{a_j\}$.

Theorem 5

For the Markov chain in Bailey's model, the matrix $K(z)$ defined in Formula (37), has m identical rows which are equal to the first row of $\hat{G}(z)$, defined in Formula (13). The vector $\underline{\gamma}$, defined in (30), is given by the first row of the matrix G. The vector $\underline{\theta}$, defined in (31), is given by

$$\underline{\theta} = \mu_1 \, \underline{e}, \tag{52}$$

where μ_1 is the first component of the vector $\underline{\mu}$. The vector $\underline{\xi}$ is given by

$$\underline{\xi} = \mu_1^{-1} \, \underline{\gamma}, \tag{53}$$

and

$$\sum_{\nu=0}^{m-1} x_\nu = \underline{\xi}\ \underline{e} = \mu_1^{-1}. \tag{54}$$

The mean $\tilde{L}$ is given by

$$\tilde{L} = \alpha^* + \frac{1}{2}(m-\alpha^*)^{-1}[\alpha^*(2) + \alpha^* + m^2(\mu_1^{-1} - 1) - \sum_{\nu=0}^{m-1} \nu^2 x_\nu]. \tag{55}$$

Proof

It is trivial to verify that $K(z)$ has identical rows. Comparing the equation (37) and the equation (14) for the first rows only, we see that these equations are the same. This shows that the rows of $K(z)$ are equal to the first row of $\hat{G}(z)$. The statement regarding $\underline{\gamma}$ is now obvious. It follows that all components of the vector $\underline{\theta}$ must be equal. Comparing the equations (21) and (31) for the first components of $\underline{\mu}$ and $\underline{\theta}$, we obtain $\theta_1 = \mu_1$, so that the equation (52) follows. Formula (53) follows from Theorem 3 and the expression (55) for $\tilde{L}$ follows by substituting (54) in (49) and simplifying.

A Markov chain of the type (1) occurs also in the study of the stationary waiting time distribution of a discrete version of the GI/G/1 queue, as investigated by G. Ponstein [14]. He considers service and interarrival distributions of lattice type. Choosing the step of the lattice as unit of time, it is assumed that the probability density of S-T (service time minus interarrival time) concentrates on the integers $\{-m,-m+1,\ldots,0,\ldots\}$ and that $P\{S-T = -m\} > 0$. Setting $P\{S-T = j\} = a_{m+j}$, for $j \geq -m$, Ponstein shows that the stationary density of the waiting time of the n-th customer is simply related to the stationary density of a Markov chain of the type which arises in Bailey's model. We refer to [14] for the details of an algorithm, which is essentially that obtained from the Rouché method. A substantial amount of work is devoted to the problems of multiple roots and in some cases,

see Ponstein's discussion in Section 6 of [14], numerical difficulties need to be overcome by trial-and-error.

Our preceding discussion provides an alternative to Ponstein's algorithm. It avoids the numerical problems of his method and appears to have also much smaller computation times.

5. Moran's Dam with Infinite Capacity

A classical model, due to Moran [5], for a dam in discrete time with discretized content, involves a Markov chain of the type (1) in which the first m rows satisfy

$$b_{j0} = \sum_{\nu=0}^{m-j} a_\nu, \quad b_{jk} = a_{k+m-j}, \quad \text{for } k \geq 1, \tag{56}$$

and $0 \leq j \leq m-1$. In this model, $\{a_k\}$ is the probability density of the number of units of water added to the dam per year and m is the maximum amount of water released at the end of each year. We assume that the capacity of the dam is infinite. A detailed discussion of this model, both for the finite and infinite capacity dam is given in Prabhu [15], Chapter 6. In Moran's model, the dam content is described immediately following releases of water. It is however easy to see that if we consider the Bailey queue, discussed in Section 4, immediately after the epochs of availability of the server, we obtain precisely the embedded Markov chain of the Moran dam.

The stationary density of the Markov chain, which is of interest here, is therefore obtained effortlessly from that discussed in Section 4. If $\{x_j'\}$ is the stationary density of the present chain, then we have

$$x_0' = \sum_{\nu=0}^{m} x_\nu, \quad x_j' = x_{m+j}, \quad \text{for } j \geq 1, \tag{57}$$

where $\{x_j\}$ is the density obtained in Section 4. The expected (stationary) dam content $\tilde{L}'$, following releases is related to the quantity $\tilde{L}$ of Formula (55) by

$$\tilde{L}' = \sum_{j=1}^{\infty} jx_j' = \tilde{L} - m + \sum_{j=0}^{m-1} (m-j)x_j = \tilde{L} - \alpha^*, \tag{58}$$

since, by Formula (48), we have

$$\sum_{j=0}^{m-1} (m-j)x_j = m - \alpha^*. \tag{59}$$

We see that Formula (59) also gives us the expected deficiency per year for the stationary dam.

Let $\eta(mr+j-1)$, for $r \geq 0$, $1 \leq j \leq m$, be the expected number of periods (years) until the first time the dam is deficient, given that the dam content at time 0+ is $mr+j-1$, then we have

Theorem 6

The mean $\eta(mr+j-1)$ is given by the j-th component of the vector

$$[I - G^{r+1} + (r+1)\tilde{G}](I - G + \tilde{G})^{-1}\underline{\mu} = \underline{\mu}^{(r+1)}. \tag{60}$$

Proof

The time until the first deficiency, starting with a dam content of $mr+j-1$, is also the first passage time from the state $(r+1,j)$ to the level 0 in a Markov chain of type (1). It follows that the vector $\underline{\mu}^{(r+1)}$ is given by

$$\underline{\mu}^{(r+1)} = \{\frac{d}{dz} \hat{G}^{r+1}(z)\}_{z=1} \underline{e} = \sum_{\nu=0}^{r} G^{\nu}\underline{\mu}, \tag{61}$$

and since

$$\sum_{\nu=0}^{r} G^{\nu}(I-G+\tilde{G}) = I - G^{r+1} + (r+1)\tilde{G}, \qquad \text{for } r \geq 0, \tag{62}$$

we obtain the stated result.

Remarks

a. The vectors $\underline{\mu}^{(r)}$ are most easily computed by the recurrence relation

$$(63) \qquad \underline{\mu}^{(1)} = \underline{\mu}, \qquad \underline{\mu}^{(r+1)} = \underline{\mu} + G\,\underline{\mu}^{(r)}, \qquad \text{for } r \geq 1.$$

b. By using formulas proved in [10], it is possible to evaluate also the second, and in principle, higher moments of this waiting time. The resulting expressions are however quite complicated and involve the matrix M.

c. We note that $\eta(mr+j-1)$ is not the expected time till the first emptiness, but the expected number of years until the dam has a positive deficiency. The formula for the expected time until the first emptiness is more complicated, but may be routinely calculated.

Corollary 5

In a dam with content $mr+j-1$ at time 0+, the expected amount $\chi(mr+j-1)$ of water released until the year of the first deficiency is given by the j-th component of the vector

$$(64) \qquad m\,\underline{\mu}^{(r+1)} + \sum_{i=1}^{m}(i-1)(G^{r+1})_{ji}.$$

Proof

The first term is obvious. The second term follows by recalling that $(G^{r+1})_{ji}$ is the conditional probability that the first deficient dam content is of size $i-1$, or equivalently that the first visit to level 0 occurs in the state $(0,i)$.

6. A Bulk Service Queue with a Threshold

We shall now consider a variant of the M/G/1 queue, which has received wide attention [3, 6, 7]. This model has a variety of concrete applications, which usually involve some interesting control aspects. We shall obtain a number of explicit and computationally tractable results not available before.

Customers arrive at a service unit according to a Poisson process of rate λ. Services occur in groups, with the group size dependent on the queue length according to the following rule. Let there be i customers waiting at the completion of a service. If $0 \leq i < L$, the server remains idle until the queue length reaches i and then starts serving all L customers. If $L \leq i \leq m$, a group of size i enters service and if $i \geq m$, a group of size m is served. It is assumed that the lengths of service of successive groups are conditionally independent, given the group sizes and that the probability distribution of the service times of a group of size j is $H_j(\cdot)$ with mean α_j, $L \leq j \leq m$. For notational purposes, we denote the Laplace-Stieltjes transform of $H_j(\cdot)$ by $h_j(s)$, Re $s \geq 0$.

As discussed in [7], the bivariate sequence of the queue lengths following departures and the times between departures defines a Markov renewal sequence, whose transition probability matrix $Q(\cdot)$ has a Laplace-Stieltjes transform $Q^*(s)$, which is of the form

$$(65) \qquad Q^*(s) = \begin{vmatrix} b_{00}(s) & b_{01}(s) & b_{02}(s) & \cdots \\ b_{10}(s) & b_{11}(s) & b_{12}(s) & \cdots \\ \vdots & \vdots & \vdots & \\ b_{m-1,0}(s) & b_{m-1,1}(s) & b_{m-1,2}(s) & \cdots \\ a_0(s) & a_1(s) & a_2(s) & \cdots \\ 0 & a_0(s) & a_1(s) & \cdots \\ 0 & 0 & a_0(s) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} .$$

The entries $b_{\nu j}(s)$, $0 \leq \nu \leq m-1$, $j \geq 0$, are given by

(66)
$$b_{\nu j}(s) = \left(\frac{\lambda}{\lambda+s}\right)^{L-\nu} \int_0^\infty e^{-(\lambda+s)u} \frac{(\lambda u)^j}{j!} dH_L(u), \quad \text{for } 0 \leq \nu \leq L,\ j \geq 0,$$
$$= \int_0^\infty e^{-(\lambda+s)u} \frac{(\lambda u)^j}{j!} dH_\nu(u), \quad \text{for } L \leq \nu \leq m-1,\ j \geq 0,$$

and the entries $a_j(s)$ are given by

(67)
$$a_j(s) = \int_0^\infty e^{-(\lambda+s)u} \frac{(\lambda u)^j}{j!} dH_m(u), \quad \text{for } j \geq 0.$$

The matrix $P = Q^*(0+)$ is clearly of the type (1) and

(68)
$$A^*(z) = h_m(\lambda-\lambda z),$$

$$B_\nu(z) = h_L(\lambda-\lambda z), \quad \text{for } 0 \leq \nu \leq L,$$
$$= h_\nu(\lambda-\lambda z), \quad \text{for } L \leq \nu \leq m-1,$$

$$\alpha^* = \lambda\ \alpha_m,$$

$$\beta^*_\nu = \lambda\ \alpha_L, \quad \text{for } 0 \leq \nu \leq L,$$
$$= \lambda\ \alpha_\nu, \quad \text{for } L \leq \nu \leq m-1.$$

Provided $\lambda\ \alpha_m < m$, the queue is stable and the stationary density $\{x_j\}$ of the queue length following departures exists. The quantities x_j can be computed by using the procedure discussed in Section 3.

A. The Queue Length in Continuous Time

In this section we investigate the stationary density of the queue length at an arbitrary point of time. Henceforth we shall only consider the stable case $\rho^* = \lambda m^{-1} \alpha_m < 1$.

We shall need to compute the fundamental mean E and the mean recurrence times E_j of the states $j \geq 0$, in the embedded Markov renewal process of the

queue. The fundamental mean is the inner product of the vector $\underline{x}$ and the vector of the row sum means of the transition matrix $Q(\cdot)$. From (65) and (66) we obtain that

$$(69) \qquad E = \frac{1}{\lambda}\sum_{\nu=0}^{L-1}(L-\nu)x_\nu + \frac{1}{\lambda}\sum_{\nu=0}^{m-1}\beta_\nu^* x_\nu + \frac{1}{\lambda}[1 - \sum_{\nu=0}^{m-1}x_\nu]\alpha^*.$$

Lemma 4

The fundamental mean E is also given by

$$(70) \qquad E = \frac{L}{\lambda}\sum_{\nu=0}^{L-1}x_\nu + \frac{m}{\lambda}[1 - \sum_{\nu=0}^{m-1}x_\nu] + \frac{1}{\lambda}\sum_{\nu=L}^{m-1}\nu x_\nu,$$

and the mean recurrence time E_j of the state j in the Markov renewal process $Q(\cdot)$ is given by

$$(71) \qquad E_j = \frac{E}{x_j}, \qquad \text{for } j \geq 0.$$

Proof

Formula (48) yields that

$$(72) \qquad \sum_{\nu=0}^{m-1}\beta_\nu^* x_\nu + \alpha^*[1 - \sum_{\nu=0}^{m-1}x_\nu] = m[1 - \sum_{\nu=0}^{m-1}x_\nu] + \sum_{\nu=0}^{m-1}\nu x_\nu,$$

and substitution in (69) leads to (70). Formula (71) is obtained by application of Thm. 6.12, p. 155 in Çinlar [2] or Thm. 2.11, p. 196 in Hunter [4].

We shall now denote the density of the queue length in continuous time by $\underline{y} = \{y_j\}$. That density is related to the density $\underline{x}$ as follows:

Theorem 7

The density $\underline{y}$ is given by

(73) $$y_j = \frac{1}{\lambda E} \sum_{\nu=0}^{j} x_\nu, \qquad \text{for } 0 \le j \le L-1,$$

$$y_j = \frac{1}{E} \sum_{\nu=0}^{L-1} x_\nu \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-L}}{(j-L)!} [1 - H_L(u)]du$$

$$+ \frac{1}{E} \sum_{\nu=L}^{j} x_\nu \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-\nu}}{(j-\nu)!} [1 - H_\nu(u)]du, \qquad \text{for } L \le \nu \le m-1,$$

$$y_j = \frac{1}{E} \sum_{\nu=0}^{L-1} x_\nu \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-L}}{(j-L)!} [1 - H_L(u)]du$$

$$+ \frac{1}{E} \sum_{\nu=L}^{m-1} x_\nu \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-\nu}}{(j-\nu)!} [1 - H_\nu(u)]du$$

$$+ \frac{1}{E} \sum_{\nu=m}^{j} x_\nu \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-\nu}}{(j-\nu)!} [1 - H_m(u)]du, \qquad \text{for } \nu \ge m.$$

The probability generating function

(74) $$Y(z) = \sum_{j=0}^{\infty} y_j z^j, \qquad \text{for } |z| \le 1,$$

is given by

(75) $$Y(z) = \frac{1}{\lambda E(1-z)} \{\sum_{\nu=0}^{L-1} x_\nu [z^\nu - z^L h_L(\lambda-\lambda z)]$$

$$+ \sum_{\nu=L}^{m-1} x_\nu z^\nu [1 - h_\nu(\lambda-\lambda z)]$$

$$+ [X(z) - \sum_{\nu=0}^{m-1} x_\nu z^\nu][1 - h_m(\lambda-\lambda z)]\}.$$

The means $\tilde{L}$ and $\tilde{L}^*$ of $\underline{x}$ and $\underline{y}$ are related by

(76) $$2\lambda E\tilde{L}^* = 2m(\tilde{L} - \sum_{\nu=0}^{m-1} \nu x_\nu) + L(L-1) \sum_{\nu=0}^{L-1} x_\nu$$

$$+ m[(m-1) - 2(m-\lambda\alpha_m)][1 - \sum_{\nu=0}^{m-1} x_\nu]$$

$$+ 2\lambda L\alpha_L \sum_{\nu=0}^{L-1} x_\nu + 2\lambda \sum_{\nu=L}^{m-1} \nu\alpha_\nu x_\nu,$$

and

(77) $$2(m-\lambda\alpha_m)(\tilde{L} - \sum_{\nu=0}^{m-1} \nu x_\nu) = \lambda^2\alpha_L^{(2)} \sum_{\nu=0}^{L-1} x_\nu + \lambda^2 \sum_{\nu=L}^{m-1} \alpha_\nu^{(2)} x_\nu$$

$$- \sum_{\nu=0}^{m-1} \nu(\nu-1)x_\nu + [\lambda^2\alpha_m^{(2)} - m(m-1) + 2m(m - \lambda\alpha_m)][1 - \sum_{\nu=0}^{m-1} x_\nu],$$

where the quantities $\alpha_\nu^{(2)}$, $L \leq \nu \leq m$, are the second moments of the probability distributions $H_\nu(\cdot)$.

Proof

The formulas (73) were proved in Neuts [7] by a classical argument based on the Key renewal theorem. The probability generating function $Y(z)$ is obtained from (73) by routine but lengthy calculations involving a few interchanges of summations and of integrations. Formula (77) is obtained by appropriate substitutions in (49) and the expression (76) is obtained by lengthy differentiations and passage to the limit in (75).

Remark

When the probability distributions $H_\nu(\cdot)$, $L \leq \nu \leq m$, are of phase type [9], the integrals which appear in the formulas (73) can be computed recursively for each ν. No numerical integrations are then required and the evaluation of the quantities y_j, $j \geq 0$, from the x_j, $j \geq 0$, is particularly easy.

Corollary 6

In the stationary version of the queue the probability P_0, that the server is idle is given by

(78) $$P_0 = \frac{\sum_{\nu=0}^{L-1} (L-\nu)\, x_\nu}{\sum_{\nu=0}^{L-1} (L-\nu)\, x_\nu + m[1 - \sum_{\nu=0}^{m-1} x_\nu] + \sum_{\nu=0}^{m-1} \nu x_\nu}.$$

Proof

$$P_0 = \sum_{j=0}^{L-1} y_j = \frac{1}{\lambda E} \sum_{j=0}^{L-1} \sum_{\nu=0}^{j} x_\nu = \frac{1}{\lambda E} \sum_{\nu=0}^{L-1} x_\nu (L-\nu). \tag{79}$$

In Formula (78), we have written λE explicitly and we have rearranged the terms so that it is clear that $0 < P_0 < 1$.

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An
A novel analysis of the steady-state probabilities of a class of infinite Markov chains is given. Markov chains of this type appear in the study of bulk queues and a variety of other stochastic models. Algorithms, which involve only real arithmetic and avoid the traditional analysis, based on Rouche's theorem, are presented.